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A New Model Selection Test with Application to the Censored Data of Carbon Nanotubes Coating

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ABSTRACT

Model selection of nano and micro droplet spreading can be widely used to predict and optimize of different coating processes such as ink jet printing, spray painting and plasma spraying. The idea of model selection is beginning with a set of data and rival models to choice the best one. The decision making on this set is an important question in statistical inference. Some tests and criteria are designed to answer to this question that which of the rival models is the best one. The purpose of this article is to propose a new interval say tracking interval for comparing the two rival models and examine its suitability in the spread data of carbon nanotubes coating. The proposed interval can be used for non-nested or nested models and whether both, one or neither is mis-specified. An important implication of the present study is that if the rival models are really close, then the proposed interval can be determined the equivalent models under censored data. Prog. Color Colorants Coat. 9 (2016), 17-28 © Institute for Color Science and Technology.

1. Introduction

Coating of a surface by droplet spreading plays an important role in several novel industrial processes, such as plasma spray coating, ink jet printing, nano safeguard coatings and nano self-assembling. Spray coating is commonly used to apply protective coatings on components to shield them from wear, corrosion, and etc. The properties of the coatings are largely dependent on the splat morphology and their stacking. Researches strongly indicate that CNTs (carbon nanotubes) play a critical role in the improvement of splat morphology [1]. Development of splat data analyzes, which can predict morphology of splats, can potentially reduce the cost of the development of new coatings considerably. The model selection will also enable us to predict, improve and optimize the design of existing spraying guns. There are different model selection tests for discriminating between two complete models. Each of the tests has advantages and disadvantages in their domain of usage. In almost all of the tests and criteria for model selection, the maximum likelihood estimator and maximized likelihood function have an essential role. For example, Gupta and Kundu [2] compared the Weibull and the generalized exponential (GE) distributions using the maximized

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likelihood ratio test. Kundu and Manglick [3] discriminated between the Log-Normal and gamma distributions, Pakyari [4] introduced the diagnostic tools based on the likelihood ratio test (LRT) and the minimum Kolmogorov distance (KD) method for discriminating between generalized exponential, geometric extreme exponential and Weibull distributions. Bromideh [5] compared the Gamma and the Log-Normal Distributions Based on Kullback-Leibler Divergence. Also Voung [6] introduced the test for comparing the two complete non-nested models. In Vuong viewpoint, the best model is the model which maximizes the relevant part of the Kullback–Leibler risk. The null hypothesis of Vuong's test is the expectation under the true model of the log-likelihood ratio (LR) of the two rival models which are equal to zero. Moreover, in many experimental studies such as plasma spray coating, it is quite common that complete data are not observed. Data obtained from such experiments are called censored data. When a data set is censored, the problem of choosing the correct distribution becomes more difficult. Because for censored data, the two models may provide similar data fit. Type I and Type II hybrid censoring schemes are the common hybrid censoring schemes. Both these censoring schemes have some disadvantages. Specifically, in Type I hybrid censoring, there may be very few or even no failures observed whereas in Type II hybrid censoring the experiment could last for a very long period of time. In order to provide a guarantee in terms of the number of failures observed as well as time to complete the test, Chandrasekar et. al. [7] introduced generalized Type II hybrid censoring scheme (GHCS) and it can be described as follows. Suppose that n identical units are put on a test, with the lifetimes and ordered lifetimes of the n items are denoted by $X_1,...,X_n$ and $Y_1,...,Y_n$ respectively. Fix $r \in \{0,...,n\}$ and time points $T_1, T_2 \in (0,\infty)$, such that $T_1 < T_2$. If r^{th} failure occurs before time T_1, the experiment terminate at T_1; if the r^{th} failure occurs between T_1 and T_2, the experiment terminate at Y_r and if the r^{th} failure occurs after time T_2, then the experiment terminate at T_2. Under this censoring scheme, it is guaranteed that the total time under test will be at most T_2. Therefore under the GHCS, we have the following three cases:

Case I:

$0 < Y_r < T_1 < T_2$ the experiment terminate at T_1

Case II:

$0 < T_1 < Y_r < T_2$ the experiment terminate at Y_r (1)

Case III:

$0 < T_1 < T_2 < Y_r$ the experiment terminate at T_2

Based on the observed data, the log-likelihood function for combined above cases can be written as:

$$L_n^f(\alpha) = \sum_{i=1}^{d} \log f^{\alpha}(y_i) + (n-d)\log \overline{F}^{\alpha}(s) \tag{2}$$

Here, $f^{\alpha}(.)$ and $\overline{F}^{\alpha}(.)$ are the probability density function and the survival function respectively. Also, d denotes the number of the total failures in experiment until time s and d_1 and d_2 denote the number of failures that occur before time points T_1 and T_2, respectively.
In other words,

$$d = \begin{cases} d_1 & :\text{caseI} \\ r & :\text{caseII} \\ d_2 & :\text{caseIII} \end{cases} \tag{3}$$

and

$$s = \begin{cases} T_1 & :\text{caseI} \\ y_r & :\text{caseII} \\ T_2 & :\text{caseIII} \end{cases} \tag{4}$$

Although some articles have been done on the generalized hybrid censoring scheme but we have not come across any article on the behavior of the two rival models under this censoring scheme for the nanotube coating data.

So, the main objective of this paper is the determination the best model for the nanotube coating data. For this purpose, first we use the asymptotic distribution of the log-likelihood ratio statistic in comparing the two rival models under GHCS. It is observed that the asymptotic distribution is normally distributed. The variance of this normal distribution can be used to construct the new model selection test say tracking interval. The tracking interval helps us to evaluate proposed models in comparison with each other. In other words, if the calculated distance includes zero, it can be concluded that based on the predetermined confidence, both proposed models are equivalent. The proposed interval is easy to compute and could be useful in a wide variety of applications. For example, Commenges et. al. [8] considered the tracking interval between two complete models in two applications. The first is a study of the relationship between body-mass index and depression in elderly people. The second is the choice between models of HIV dynamics, where one model makes the distinction between activated CD4+T lymphocytes and the other does not. Panahi and Sayyareh [9-11] used the tracking interval for comparison of two rival models of micro-droplet splashing data under different censoring schemes.

The second objective of this paper is to analyze the carbon nanotubes coating data. We consider a large class of probabilistic models. Then we construct the tracking intervals to compare the two rival models under different censoring schemes. The rest of the paper is organized as follows. In Section 2, using the asymptotic distribution of LR Statistic, we propose the tracking interval for the difference of the expected Kullback–Leibler (KL) divergence of two rival models under generalized Type II hybrid censoring scheme. Analysis of the splats reinforced with carbon nanotubes data are provided in Section 3 and finally we conclude the paper in Section 4.

2. The New Test for Comparison the Two Models

Consider a sample of independently identically distributed (*i.i.d.*) random variables, $X_1,..., X_n$, having probability density function $h(x)=h$. Let us consider two rival models, $F^{\alpha}=\left\{f^{\alpha}(.);\ \alpha\in \mathrm{M}\right\}=(f)$ and $G^{\beta}=\left\{g^{\beta}(.);\ \beta\in \mathrm{B}\right\}=(g)$, where, M and B are the parameter spaces of α and β respectively.

Definition: (f) is well specified if there is a true value $\alpha_0\in \mathrm{M}$ such that $f^{\alpha_0}(.)=h$; otherwise it is mis-specified. Now, based on the following assumptions, we first provide the asymptotic distribution of the LR statistic and then construct the tracking interval.

The minimum assumptions, $\Re$, for non-degenerate interval M are:

$\Re_1$: For almost all x, the derivatives $(\partial/\partial\alpha)\ln f^{\alpha}(x)$ and $(\partial^2/\partial\alpha^2)\ln f^{\alpha}(x)$ all exist for every $\alpha\in M$.

$\Re_2$: For all $\alpha\in M$, the partial derivative $(\partial/\partial\alpha)f^{\alpha}(x)$, is integrable, the partial derivative $(\partial/\partial\alpha)F^{\alpha}(x)$, exists for $x\in\chi$, ; and satisfies,

$$(\partial/\partial\alpha)F^{\alpha}(x)=\int_{-\infty}^{x}(\partial/\partial\alpha)f^{\alpha}(u)\,du$$

$\Re_3$: For every α, we have,

$$\left|\frac{\partial}{\partial\alpha}f^{\alpha}(x)\right|\le \mathrm{K}_1,\quad \left|\frac{\partial^2}{\partial\alpha^2}f^{\alpha}(x)\right|\le \mathrm{K}_2$$

$$\text{and}\quad \left|\frac{\partial^3}{\partial\alpha^3}f^{\alpha}(x)\right|\le \mathrm{K}_3\ ;$$

where, $\int \mathrm{K}_i\, d\mu(x)<\infty$; $i=1,2,3$. (μ is the Lebesgue measure and $\alpha\in M$).

$\Re_4$: For every $\alpha\in M$, $\dfrac{1}{\overline{F}^{\alpha}(x)}$ is bounded by $\upsilon(x)$ respectively, where, $E(\upsilon(X))\le C$; C is positive constant, (notice that, $E(\upsilon(X))=\int \upsilon(x)f^{\alpha}(x)d\mu(x)$).

$\Re_5$: For every α, we have,

$$\wp=\int\left(\frac{\partial}{\partial\alpha}\ln f^{\alpha}(x)\right)^2 f^{\alpha}(x)d\mu(x)<\infty\,.$$

2.1. Asymptotic Distribution of the LR Statistic

From (2), the difference of the log-likelihood functions of the two rival models under GHCS can be obtained as:

$$L_n^{f/g}(\hat{\alpha}_n,\hat{\beta}_n)=L_n^f(\hat{\alpha}_n)-L_n^g(\hat{\beta}_n)$$

$$=\sum_{i=1}^{d}\log\frac{f^{\hat{\alpha}_n}(y_i)}{g^{\hat{\beta}_n}(y_i)}+(n-d)\log\frac{\overline{F}^{\hat{\alpha}_n}(s)}{\overline{G}^{\hat{\beta}_n}(s)} \qquad (5)$$

where, $\hat{\alpha}_n$ and $\hat{\beta}_n$ are the maximum likelihood estimators for the parameters α and β respectively.

Lemma: Suppose that $Y_1,...,Y_d$ are distributed as the order statistic of a random sample of size d from truncated distribution at s by probability density function (pdf) h^*. Now, if $\frac{r}{n}\to p$ as $n\to\infty$ such that $Y_r\xrightarrow{P}\zeta_p$, the p^{th} percentile of true distribution respectively, then from Voung [6] and the property of Continuous Mapping, we have

$$\frac{1}{n}\sum_{i=1}^{d}\log\frac{f^{\hat{\alpha}_n}(y_i)}{g^{\hat{\beta}_n}(y_i)}\xrightarrow{P}\breve{p}E_{h^*}\left[\log\frac{f^{\alpha_*}(Y)}{g^{\beta_*}(Y)}\right];\text{and}$$

$$\frac{1}{n}(n-d)\log\frac{\overline{F}^{\hat{\alpha}_n}(s)}{\overline{G}^{\hat{\beta}_n}(s)}\xrightarrow{P}(1-\breve{p})\log\frac{\overline{F}^{\alpha_*}(\breve{\zeta}_p)}{\overline{G}^{\beta_*}(\breve{\zeta}_p)},$$

where, d and s are defined in (3) and (4) respectively and

$$\breve{p}=\lim_{n\to\infty}\frac{d}{n}=\begin{cases}\lim\limits_{n\to\infty}\frac{d_1}{n} & if \quad d=d_1\\ \lim\limits_{n\to\infty}\frac{r}{n} & if \quad d=r\\ \lim\limits_{n\to\infty}\frac{d_2}{n} & if \quad d=d_2\end{cases} \qquad \&$$

$$\breve{\zeta}_p=\begin{cases}\zeta_{F(T_1)}; & if \quad F(T_1)>p\\ \zeta_p; & if \quad F(T_1)<p<F(T_2)\\ \zeta_{F(T_2)}; & if \quad F(T_1)<F(T_2)<p\end{cases}$$

Then the difference log-likelihood function of the two mis-specified rival models under generalized Type II hybrid censored data is converges in probability as below:

$$\frac{1}{n}L_n^{f/g}\left(\hat{\alpha}_n,\hat{\beta}_n\right)=L_n^f(\hat{\alpha}_n)-L_n^g(\hat{\beta}_n)$$

$$\xrightarrow{P}\left\{\breve{p}E_{h^*}\left[\log\frac{f^{\alpha_*}(Y)}{g^{\beta_*}(Y)}\right]+\left(1-\breve{p}\right)\log\frac{\overline{F}^{\alpha_*}(\breve{\zeta}_p)}{\overline{G}^{\beta_*}(\breve{\zeta}_p)}\right\}$$

where,

$$L_n^f(\hat{\alpha}_n)-L_n^g(\hat{\beta}_n)$$

$$=\sum_{i=1}^{d}\log\frac{f^{\hat{\alpha}_n}(y_i)}{g^{\hat{\beta}_n}(y_i)}+(n-d)\log\frac{\overline{F}^{\hat{\alpha}_n}(s)}{\overline{G}^{\hat{\beta}_n}(s)}$$

and

$$\alpha_*=\arg\max_{\alpha\in\mathrm{M}}\left\{\breve{p}E_{h^*}\left[\log f^{\alpha}(Y)\right]+\left(1-\breve{p}\right)\log\overline{F}^{\alpha}(\breve{\zeta}_p)\right\}$$

$$\beta_*=\arg\max_{\beta\in\mathrm{B}}\left\{\breve{p}E_{h^*}\left[\log g^{\beta}(Y)\right]+\left(1-\breve{p}\right)\log\overline{G}^{\beta}(\breve{\zeta}_p)\right\}$$

are pseudo-true values of α and β, respectively.

Theorem 2 (Asymptotic Distribution of the LR Statistic, $L_n^{f/g}(\hat{\alpha}_n,\hat{\beta}_n)$**):** Under regularity conditions $\Re_1-\Re_5$, suppose that the proposed model is mis-specified and $f^{\alpha^*}\neq g^{\beta^*}$, then,

$$\sqrt{n}\left(\frac{1}{n}L_n^{f/g}\left(\hat{\alpha}_n,\hat{\beta}_n\right)-\breve{p}E_{h^*}\left[\log\frac{f^{\alpha_*}(Y)}{g^{\beta_*}(Y)}\right]-\left(1-\breve{p}\right)\log\frac{\overline{F}^{\alpha_*}(\breve{\zeta}_p)}{\overline{G}^{\beta_*}(\breve{\zeta}_p)}\right)\xrightarrow{D}N\left(0,\omega_{*GHCS}^2\right) \qquad (6)$$

where,

$$\omega^2_{*GHCS} = Var_h\left(\log\frac{f^{\alpha_*}(W)}{g^{\beta_*}(W)}\right) + (1-\breve{p})Var_{h_1^*}\left(\log\frac{f^{\alpha_*}(Z)}{g^{\beta_*}(Z)}\right) \quad (7)$$

and $W=(w_1,...,w_n)=$ the complete data and $Z=(z_1,...,z_{n-d})=$ the complete data of size $n-d$, from the left truncated population with density function, $h_1^* = \frac{f^{\alpha}(z)}{\overline{F}^{\alpha}(s)}; \quad z>s$. Note that, the sequences of random variables W's and Z's are independent.

Proof: Using the missing information principles of Louis [12], the observed information under GHCS is:

$$\sum_{i=1}^{d}\log f^{\alpha}(y_i) = \sum_{i=1}^{n}\log f^{\alpha}(w_i) - \sum_{i=1}^{n-d}\log f^{\alpha}(z_i \mid Y) \quad (8)$$

where, W and Z are defined in (7). For simplicity, we replace $f^{\alpha}(z_i \mid Y)$ by $f^{\alpha}(z_i)$ throughout the proof. Now, from the Voung [6], we can write

$$L_n^f(\hat{\alpha}_n) = L_n^f(\alpha_*) - \frac{n}{2}(\hat{\alpha}_n-\alpha_*)' J_{f_{GHCS}}(\hat{\alpha}_n-\alpha_*) + o_p(1)$$

and

$$L_n^g(\hat{\beta}_n) = L_n^f(\beta_*) - \frac{n}{2}(\hat{\beta}_n-\beta_*)' J_{g_{GHCS}}(\hat{\beta}_n-\beta_*) + o_p(1)$$

Note that, $n^{-1}\frac{\partial^2 L_n^f(\hat{\alpha}_n)}{\partial\alpha\partial\alpha'} \xrightarrow{P} -J_{f_{GHCS}}$ and similarly, $n^{-1}\frac{\partial^2 L_n^g(\hat{\beta}_n)}{\partial\beta\partial\beta'} \xrightarrow{P} -J_{g_{GHCS}}$. Also, it is known that, $\sqrt{n}(\hat{\alpha}_n-\alpha_*)$ and $\sqrt{n}(\hat{\beta}_n-\beta_*)$ are $O_p(1)$ (see appendix). So, we have

$$\sqrt{n}\left(\frac{1}{n}L_n^{f/g}\left(\hat{\alpha}_n,\hat{\beta}_n\right) - \breve{p}E_{h^*}\left[\log\frac{f^{\alpha_*}(Y)}{g^{\beta_*}(Y)}\right] - (1-\breve{p})\log\frac{\overline{F}^{\alpha_*}(\breve{\zeta}_p)}{\overline{G}^{\beta_*}(\breve{\zeta}_p)}\right) = \sqrt{n}\left\{\frac{1}{n}L_n^{f/g}(\alpha_*,\beta_*) - \breve{p}E_{h^*}\left[\log\frac{f^{\alpha_*}(Y)}{g^{\beta_*}(Y)}\right] - (1-\breve{p})\log\frac{\overline{F}^{\alpha_*}(\breve{\zeta}_p)}{\overline{G}^{\beta_*}(\breve{\zeta}_p)}\right\} + o_p(1)$$

But from the multivariate central Theorem, the first term in the right hand side converges in distribution to $N(0,\omega^2_{*GHCS})$. It now suffices to show that

$$\omega^2_{*GHCS} = Var_h\left(\log\frac{f^{\alpha_*}(W)}{g^{\beta_*}(W)}\right) + (1-\breve{p})Var_{h_1*}\left(\log\frac{f^{\alpha_*}(Z)}{g^{\beta_*}(Z)}\right)$$

Now, using the missing information principle (8), we can write

$$\omega^2_{*UHCS} = \frac{1}{n}Var\left(\sum_{i=1}^{d}\log\frac{f^{\alpha_*}(Y_i)}{g^{\beta_*}(Y_i)} + (n-d)\log\frac{\overline{F}^{\alpha_*}(s)}{\overline{G}^{\beta_*}(s)}\right)$$

$$= \frac{1}{n}Var\left[\left(\sum_{i=1}^{n}\log\frac{f^{\alpha_*}(W_i)}{g^{\beta_*}(W_i)} - \sum_{i=1}^{n-d}\log\frac{f^{\alpha_*}(Z_i)}{g^{\beta_*}(Z_i)} + (n-d)\log\frac{\overline{F}^{\alpha_*}(s)}{\overline{G}^{\beta_*}(s)}\right)\right]$$

Now, If, $\frac{n-d}{n} \to 1-\breve{p}$ as $n \to \infty$ such that $s \to \breve{\xi}_p$ in probability, then using Continuous Mapping Theorem and the properties of variance, we have

$$\omega^2_{*GHCS} = Var_h\left(\log \frac{f^{\alpha_*}(W)}{g^{\beta_*}(W)}\right) + (1-\breve{p})Var_{h_1*}\left(\log \frac{f^{\alpha_*}(Z)}{g^{\beta_*}(Z)}\right).$$

Note that, we can propose the following empirical variance for constructing the useful interval. This interval can be used for comparing two models in the applied sciences.

$$\hat{\omega}^2_{GHCS} = \frac{1}{n}\sum_{i=1}^{n}\left(\log \frac{f^{\hat{\alpha}_n}(w_i)}{g^{\hat{\beta}_n}(w_i)}\right)^2 - \left(\frac{1}{n}\sum_{i=1}^{n}\left(\log \frac{f^{\hat{\alpha}_n}(w_i)}{g^{\hat{\beta}_n}(w_i)}\right)\right)^2 + (1-\frac{d}{n})\left[\frac{1}{n-d}\sum_{i=1}^{n-d}\left(\log \frac{f^{\hat{\alpha}_n}(z_i)}{g^{\hat{\beta}_n}(z_i)}\right)^2 - \left(\frac{1}{n-d}\sum_{i=1}^{n-d}\left(\log \frac{f^{\hat{\alpha}_n}(z_i)}{g^{\hat{\beta}_n}(z_i)}\right)\right)^2\right] \quad (9)$$

2.2. Tracking Interval for Comparing the Two Models under GHCS

In this section we propose the model selection test using an interval say tracking interval instead of hypothesis testing of Voung as its dual; it is because the confidence interval is a set of all acceptable hypotheses with pre-assigned confidence.

We propose the tracking interval for a difference of expected Kullback-Leibler risks, $\Delta_{hybrid}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) = EKL(h, f^{\hat{\alpha}_n}) - EKL(h, g^{\hat{\beta}_n})$, to compare the two rival models based on GHCS, where

$$EKL(h, f^{\hat{\alpha}_n}) = KL(h, f^{\alpha_*}) + \frac{1}{2n} Tr\left(I_{f_{GHCS}} J^{-1}_{f_{GHCS}}\right) + o(n^{-1})$$

and $KL(h, f^{\alpha_*}) = E_h\left(\log \frac{h(X)}{f^{\alpha_*}(X)}\right)$ &

$$I_{f_{GHCS}} = E_h\left(\frac{\partial \ln f^{\alpha}(Y)}{\partial \alpha} . \frac{\partial \ln f^{\alpha}(Y)}{\partial \alpha'}\right)\Bigg|_{\alpha_*}.$$

This interval has another interpretation for the use of Akaike information criterion (AIC). In fact we are not in a situation to detect the best model but we are in search for a model which has the relatively less risk compared to other models. Now, using (9) and Panahi and Sayyareh [9] [11], the tracking interval under GHCS is given by

$$\left[D_{GHCS}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) - n^{-1/2} z_{\alpha/2}\hat{\omega}_{GHCS},\ D_{GHCS}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) + n^{-1/2} z_{\alpha/2}\hat{\omega}_{GHCS}\right] \quad (10)$$

where,

$$D_{GHCS}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) = -\frac{1}{n}\left[L_n^{f/g}(\hat{\alpha}_n, \hat{\beta}_n) - (p-q)\right]$$

; p and q are the number of parameters in two models. This interval has the property as

$$P_h\left[A_n < \Delta_{GHCS}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) < B_n\right] \to 1-\alpha$$

where,

$$A_n = D_{GHCS}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) - n^{-1/2} z_{\alpha/2}\hat{\omega}_{GHCS} \ ;$$

$$B_n = D_{GHCS}(f^{\hat{\alpha}_n}, g^{\hat{\beta}_n}) + n^{-1/2} z_{\alpha/2}\hat{\omega}_{GHCS}$$

and P_h represents the probability with density h. Based on this tracking interval, if the calculated distance (10) includes zero, it can be concluded that based on the predetermined confidence both the proposed models are equivalent. An interval which does not contain zero indicates that one model is better than the other one.

3. Applications of Tracking Interval to Real Data

In this section, we analyze the data of splats with carbon nanotube (CNT) addition, obtained in Keshri and Agarwal [1]. The data sets consist the sub-micron Al2O3 powder was spray dried (referred as A-SD), sub-micron Al2O3 with 4 weight percent of CNTs (referred as A4C-SD) and 8 weight percent of CNTs (referred as A8C-SD) materials. First we want to choose the best fitted model based on the current criteria such as log-likelihood (LL) values, Akaike information criterion (AIC) values, Bayesian information criterion (BIC) values and the Kolmogorov-Smirnov (K-S) distances. It is observed that the mentioned data sets are always positive and therefore, it is reasonable to analyze the data of splats reinforced with carbon nanotubes using the probability distributions, which have support only on the positive real axis. Thus, we fit different distribution functions, namely generalized exponential (GE),

$$f_{GE}^{(\alpha,\beta)}(x) = \alpha\beta e^{-\beta x}(1-e^{-\beta x})^{\alpha-1}$$

exponentiated Weibull (EW),

$$f_{EW}^{(\alpha,\beta,\gamma)}(x) = \alpha\beta\gamma x^{b-1} e^{-px^b}(1-e^{-px^b})^{\gamma-1}$$

Burr III (BIII),

$$f_{BIII}^{(\alpha,\beta)}(x) = \alpha\beta x^{-\beta-1}(1+x^{-\beta})^{-\alpha-1}$$

Weibull (W),

$$f_{W}^{(\alpha,\beta)}(x) = \alpha\beta x^{b-1} e^{-px^b}$$

inverse Weibull (IW),

$$f_{IW}^{(\alpha,\beta)}(x) = \alpha\beta x^{-b-1} e^{-px^{-b}}$$

Burr XII (BXII)

$$f_{BXII}^{(\alpha,\beta)}(x) = \alpha\beta x^{\beta-1}(1+x^{\beta})^{-\alpha-1}$$

and exponentiated Burr III (EBIII)

$$f_{EBIII}^{(\alpha,\beta,\gamma)}(x) = \alpha\beta\gamma x^{-\beta-1}(1+x^{-\beta})^{-\alpha\gamma-1}$$

and report the estimated parameter values, K-S distances, AICs, BICs and LL values in Tables 1, 2 and 3 respectively. From the Tables 1-3, it is clear that, generalized exponential (GE) distribution is the best fitted model based on maximum log-likelihood values, minimum AIC and BIC values or the minimum K-S distance. Now using the tracking interval we want to observe how the two models behave for these data sets. For A-SD, we consider the following three different cases of censoring schemes:

Case 1: $T_1 = 29.25$, $T_2 = 29.75$ and r=22($y_r = 27.25$).

Case 2: $T_1 = 27.75$, $T_2 = 30.75$ and r=46($y_r = 28.25$).

Case 3: $T_1 = 27.75$, $T_2 = 30.75$ and r=86($y_r = 31.25$).

For all cases of censoring schemes, we consider two different cases of rival models:

A: GE (f) and EW (g) distributions (GE and EW are well specified and mis- specified models respectively).

B: BIII (f) and IW (g) distributions (Two mis-specified models).

Furthermore, for A4C-SD and ABC-SD, we also adopt the three different cases of censoring schemes as:

Case 1: $T_1 = 34.25$, $T_2 = 35.75$ and r=30($y_r = 33.75$).

Case 2: $T_1 = 33.25$, $T_2 = 36.25$ and r=61 ($y_r = 35.75$).

Case 3: $T_1 = 33.25$, $T_2 = 36.25$ and r=87($y_r = 37.25$).

and

Case 1: $T_1 = 43.25$, $T_2 = 45.25$ and r=34($y_r = 41.25$).

Case 2: $T_1 = 42.75$, $T_2 = 45.25$ and r=66 ($y_r = 44.25$).

Case 3: $T_1 = 42.75$, $T_2 = 44.25$ and r=86($y_r = 45.25$).

respectively. Similar to A-SD, we consider two different cases of rival models:

A: GE and EW distributions (GE and EW are well specified and mis- specified models respectively).

B: IW and BIII distributions (Two mis-specified models).

In all the three cases, we have estimated the unknown parameters using the MLEs and then constructed the tracking intervals. The results are reported in Table 4. First, we consider EW and GE distributions as the rival models (A). For all cases and materials, it is observed that both limits of the tracking intervals are negative, which indicates that the GE is better than the EW to estimate the true model (as we expected). Also, for B, the both limits of the tracking intervals are positive, which indicates that the IW is better than the BIII density to estimate the true model for all cases and materials. Furthermore in all the cases, the lengths of the tracking intervals are small. Thus the two models are similar in information criteria sense. So, the tracking interval for the difference of risks is easy to compute and could be useful in a wide variety of applications.

Table 1: Estimated parameters, K-S distances and AIC values for different distribution functions of A-SD.

Distribution	Estimated parameters			K-S	AIC	BIC	LL
GE	$\alpha = 3.551\times10^{9}$	$\beta = 0.7061$	-	0.1005	3.290×10^{2}	3.340×10^{2}	-1.625×10^{2}
EW	$\alpha = 1.208\times10^{-5}$	$\beta = 3.8572$	$\gamma = 1.019\times10^{2}$	0.1293	3.316×10^{2}	3.349×10^{2}	-1.627×10^{2}
BIII	$\alpha = 9.908\times10^{9}$	$\beta = 6.8978$	-	0.3622	4.510×10^{2}	4.560×10^{2}	-2.235×10^{2}
W	$\alpha = 5.383\times10^{-4}$	$\beta = 2.2335$	-	0.5486	6.442×10^{2}	6.492×10^{2}	-3.201×10^{2}
IW	$\alpha = 1.75\times10^{11}$	$\beta = 7.7463$	-	0.3521	4.325×10^{2}	4.375×10^{2}	-2.142×10^{2}
BXII	$\alpha = 0.07332$	$\beta = 4.19915$	-	0.6226	10.055×10^{2}	10.105×10^{2}	-5.007×10^{2}
EBIII	$\alpha = 1.014\times10^{7}$	$\beta = 9.7985$	$\gamma = 1.510\times10^{7}$	0.2816	4.000×10^{2}	4.030×10^{2}	-1.680×10^{3}

Table 2: Estimated parameters, K-S distances and AIC values for different distribution functions of A4C-SD**.**

Distribution	Estimated parameters			K-S	AIC	BIC	LL
GE	$\alpha = 3.398\times10^{9}$	$\beta = 0.6435$	-	0.1125	3.422×10^{2}	3.472×10^{2}	-1.691×10^{2}
EW	$\alpha = 1.125\times10^{-6}$	$\beta = 4.1972$	$\gamma = 2.068\times10$	0.2099	3.877×10^{2}	3.907×10^{2}	-1.908×10^{2}
BIII	$\alpha = 3.740\times10^{10}$	$\beta = 6.8785$	-	0.4311	4.935×10^{2}	4.804×10^{2}	-2.447×10^{2}
W	$\alpha = 6.163\times10^{-4}$	$\beta = 2.0969$	-	0.5925	7.054×10^{2}	7.104×10^{2}	-3.507×10^{2}
IW	$\alpha = 2.760\times10^{12}$	$\beta = 8.1016$	-	0.3919	4.663×10^{2}	4.714×10^{2}	-2.312×10^{2}
BXII	$\alpha = 0.08162$	$\beta = 3.45004$	-	0.6240	10.737×10^{2}	10.787×10^{2}	-5.348×10^{2}
EBIII	$\alpha = 1.080\times10^{7}$	$\beta = 9.2458$	$\gamma = 1.498\times10^{7}$	0.2797	4.457×10^{2}	4.487×10^{2}	-1.739×10^{2}

Table 3: Estimated parameters, K-S distances and AIC values for different distribution functions of A8C-SD.

Distribution	Estimated parameters			K-S	AIC	BIC	LL
GE	$\alpha = 1.124\times10^{9}$	$\beta = 0.48833$	-	0.2285	3.580×10^{2}	3.631×10^{2}	-1.770×10^{2}
EW	$\alpha = 3.591\times10^{-7}$	$\beta = 4.3474$	$\gamma = 7.719\times10$	0.2201	3.622×10^{2}	3.672×10^{2}	-1.791×10^{2}
BIII	$\alpha = 3.873\times10^{10}$	$\beta = 6.5134$	-	0.4366	5.416×10^{2}	5.466×10^{2}	-2.688×10^{2}
W	$\alpha = 0.19341$	$\beta = 0.4094$	-	0.5845	10.463×10^{2}	10.514×10^{2}	-5.212×10^{2}
IW	$\alpha = 8.003\times10^{11}$	$\beta = 7.3139$	-	0.4262	5.208×10^{2}	5.259×10^{2}	-2.584×10^{2}
BXII	$\alpha = 0.0720$	$\beta = 3.6803$	-	0.6249	11.252×10^{2}	11.302×10^{2}	-5.606×10^{2}
EBIII	$\alpha = 4.998\times10^{6}$	$\beta = 8.6464$	$\gamma = 2.495\times10^{7}$	0.4145	3.624×10^{3}	3.629×10^{3}	-1.810×10^{3}

Table 4: Tracking intervals for two rival models (A and B) and three censoring schemes (three cases).

Cases	Case 1		Case 2		Case 3	
Rival models	*A*	*B*	*A*	*B*	*A*	*B*
A-SD						
Lower	-1.342864	3.580841	-1.155680	2.04677	-8.172×10^{-1}	5.607175
Upper	-1.057849	3.644199	-8.124×10^{-1}	2.110685	-6.343×10^{-1}	5.669738
Length	2.850×10^{-1}	0.063358	3.433×10^{-1}	0.063914	1.829×10^{-1}	0.062563
A4C-SD						
Lower	-9.210×10^{-1}	1.695464	-1.019408	3.724907	-1.125662	5.225219
Upper	-6.858×10^{-1}	1.748430	-7.988×10^{-1}	3.777338	-9.601×10^{-1}	5.277229
Length	2.351×10^{-1}	0.052965	2.205×10^{-1}	0.052430	1.655×10^{-1}	0.052009
A8C-SD						
Lower	-1.229083	3.30792	-1.243391	4.655653	-1.420072	5.452311
Upper	-1.092203	3.34547	-1.130999	4.692975	-1.309900	5.489492
Length	1.368×10^{-1}	0.03755	1.123×10^{-1}	0.037322	1.101×10^{-1}	0.037180

4. Conclusions

In this paper we consider the problem of comparing the two rival models when the data are generalized hybrid censored sample of carbon nanotubes coating. We drive the asymptotic distribution of the log-likelihood ratio (LR) statistic under GHCS. The results established will provide insight into the missing information principle. Then, using the asymptotic variance of this statistic, we introduce the new interval say tracking interval for model selection. A real example has been presented to illustrate all the inferential results established here. Based on the limited set of data and using several statistical criteria such as minimum K-S distance, minimum AIC and BIC values and maximum LL value, the GE distribution function appears to be more appropriate statistical distribution function. Moreover the results indicate that the tracking interval works quite well in discriminating between the two rival models and has advantage instead of other model selection tests. We hope that the new model selection test will attract wider application in all areas of research.

Appendix

From The Taylor expansion of $n^{-1}\frac{\partial L_n^f(\alpha)}{\partial \alpha}$ around $\alpha=\alpha_0$ gives (For simplicity, we consider α as a single parameter of interest):

$$n^{-1}\frac{\partial L_n^f(\alpha)}{\partial \alpha} = n^{-1}\frac{\partial L_n^f(\alpha)}{\partial \alpha}\Big|_{\alpha=\alpha_0} + n^{-1}(\alpha-\alpha_0)\frac{\partial^2 L_n^f(\alpha)}{\partial \alpha^2}\Big|_{\alpha=\alpha_0} + o_p(1) \quad \text{(A.1)}$$

$$= A_1 + A_2(\alpha-\alpha_0) + o_p(1)$$

where,

$$A_1 = \frac{1}{n}\left\{(\sum_{i=1}^{d}\frac{\partial}{\partial \alpha}\log f^{\alpha}(y_i)) + (n-d)\frac{\partial}{\partial \alpha}\log(\overline{F}^{\alpha}(s))\right\}_{\alpha=\alpha_0}$$

and $$A_2 = \frac{1}{n}\left\{(\sum_{i=1}^{d}\frac{\partial^2}{\partial \alpha^2}\log f^{\alpha}(y_i)) + (n-d)\frac{\partial^2}{\partial \alpha^2}\log(\overline{F}^{\alpha}(s))\right\}_{\alpha=\alpha_0}$$

We will show that,

$A_1 \xrightarrow{P} 0$ and $A_2 \xrightarrow{P} -J_{f_{GHCS}}$, where $J_{f_{GHCS}}$ is constant. From (16), A_1 can be rewritten as

$$A_1 = \frac{1}{n}\left\{(\sum_{i=1}^{n}\frac{\partial}{\partial \alpha_0}\log f^{\alpha}(w_i)) - \sum_{i=1}^{n-d}\frac{\partial}{\partial \alpha_0}\log f^{\alpha}(z_i) + (n-d)\frac{\partial}{\partial \alpha_0}\log(\overline{F}^{\alpha}(s))\right\} \equiv \frac{1}{n}(A_1^{*}-A_1^{**}) \quad \text{(A.2)}$$

where, $\frac{\partial}{\partial \alpha_0}\log f^{\alpha}(.)$ means that $\frac{\partial}{\partial \alpha_0}\log f^{\alpha}(.)\Big|\alpha=\alpha_0$. So, from Cramér [13], $\frac{1}{n}A_1^{*} = \frac{1}{n}\sum_{i=1}^{n}\frac{\partial}{\partial \alpha_0}\log f^{\alpha}(w_i) \xrightarrow{P} 0$ and we will prove that,

$$\frac{1}{n}A_1^{**} = \frac{1}{n}\sum_{i=1}^{n-d}\frac{\partial}{\partial \alpha_0}\log f^{\alpha}(z_i) - (n-d)\frac{\partial}{\partial \alpha_0}\log(\overline{F}^{\alpha}(s)) \xrightarrow{P} 0$$

We can rewritten A_1^{**} as,

$$A_1^{**} = \sum_{i=1}^{n-d}\frac{\partial}{\partial \alpha_0}\log f^{\alpha}(z_i) - \sum_{i=1}^{n-d}E\left(\frac{\partial}{\partial \alpha_0}\log f^{\alpha}(Z_i)\right) + \sum_{i=1}^{n-d}E\left(\frac{\partial}{\partial \alpha_0}\log f^{\alpha}(Z_i)\right) - (n-d)\frac{\partial}{\partial \alpha_0}\log(\overline{F}^{\alpha}(s))$$

.

So we have,

$$E\left(\frac{\partial}{\partial \alpha_0}\log f^{\alpha}(Z)\right) = \int_s^{\infty}\left(\frac{\partial}{\partial \alpha_0}\log f^{\alpha}(z)\right)\frac{f^{\alpha}(z)}{\overline{F}^{\alpha}(s)}\,d\mu(z) \quad \text{(A.3)}$$

$$= \frac{\frac{\partial}{\partial \alpha_0}\overline{F}^{\alpha}(s)}{\overline{F}^{\alpha}(s)} = \frac{\partial}{\partial \alpha_0}\log(\overline{F}^{\alpha}(s))$$

Thus, $\frac{A_1^{**}}{n} \xrightarrow{P} 0$. Now, by using Slutsky's Theorem, the result follows ($A_1 \xrightarrow{P} 0$). Similarly, we consider, $A_2 = \frac{1}{n}(A_2^{*}-A_2^{**})$, where, $A_2^{*} = \sum_{i=1}^{n}\frac{\partial^2}{\partial \alpha_0^2}\log f^{\alpha}(w_i)$ and

$$A_2^{**} = \sum_{i=1}^{n-d}\frac{\partial^2}{\partial \alpha_0^2}\log f^{\alpha}(z_i) - (n-d)\frac{\partial^2}{\partial \alpha_0^2}\log(\overline{F}^{\alpha}(s))$$

We know that, $\frac{A_2^{*}}{n} \xrightarrow{P} -\wp$ (defined in $\Re_5$) and,

$$\frac{A_2^{**}}{n}=\frac{n-d}{n}\left\{\frac{1}{n-d}\right.$$

$$\times\left(\sum_{i=1}^{n-d}\frac{\partial^2}{\partial\alpha_0^2}\log f^{\alpha}(z_i)-\sum_{i=1}^{n-d}E\left(\frac{\partial^2}{\partial\alpha_0^2}\log f^{\alpha}(Z_i)\right)\right)\Bigg\} \quad \text{(A.4)}$$

$$-\frac{1}{n}\left\{(n-d)\frac{\partial^2}{\partial\alpha_0^2}\log(\overline{F}^{\alpha}(s))\right.$$

$$\left.-\sum_{i=1}^{n-d}E\left(\frac{\partial^2}{\partial\alpha_0^2}\log f^{\alpha}(Z_i)\right)\right\}$$

The first term in (A.4) converges in probability to zero. So, based on (A.3) and after some simplification, we obtain

$$\frac{\partial^2}{\partial\alpha_0^2}\log\overline{F}^{\alpha}(s)=\frac{\frac{\partial^2}{\partial\alpha_0^2}\overline{F}^{\alpha}(s)}{\overline{F}^{\alpha}(s)} \quad \text{(A.5)}$$

$$-\left[E\left(\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(Z)\right)\right]^2,$$

and

$$E\left(\frac{\partial^2}{\partial\alpha_0^2}\log f^{\alpha}(Z)\right)=$$

$$\int_s^{\infty}\left[\left(\frac{\partial^2}{\partial\alpha_0^2}\log f^{\alpha}(z)/f^{\alpha}(z)\right)-\left(\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(z)\right)^2\right] \quad \text{(A.6)}$$

$$\times\frac{f^{\alpha}(z)}{\overline{F}^{\alpha}(s)}d\mu(z)=\frac{\frac{\partial^2}{\partial\alpha_0^2}\overline{F}^{\alpha}(s)}{\overline{F}^{\alpha}(s)}$$

$$-\int_s^{\infty}\left(\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(z)\right)^2\frac{f^{\alpha}(z)}{\overline{F}^{\alpha}(s)}d\mu(z).$$

Thus, from (A.4) - (A.6), we have

$$\frac{1}{n-d}\sum_{i=1}^{n-d}\left\{\frac{\partial^2}{\partial\alpha_0^2}\log(\overline{F}^{\alpha}(s))-E\left(\frac{\partial^2}{\partial\alpha_0^2}\log f^{\alpha}(Z_i)\right)\right\} \quad \text{(A.7)}$$

$$=\frac{1}{n-d}\sum_{i=1}^{n-d}Var\left(\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(Z_i)\right)=B^*$$

where, B^* converges to bounded value, say ζ. Thus, $-\frac{A_2^{**}}{n}\xrightarrow{P}(1-\breve{p})\zeta$, Now, combining of this results gives, $A_2=\frac{1}{n}(A_2^*-A_2^{**})\xrightarrow{P}-J_{f_{GHCS}}$, where, $J_{f_{GHCS}}\equiv\wp+(1-\breve{p})\zeta$. Now, from (A.1) and (A.2), we have

$$\sqrt{nJ_{f_{GHCS}}}(\hat{\alpha}_n-\alpha_0)=\frac{\sqrt{n}A_1/\sqrt{J_{f_{GHCS}}}}{-A_2/J_{f_{GHCS}}}$$

$$=\left(nJ_{f_{GHCS}}\right)^{-1/2}\left(-A_2/J_{f_{GHCS}}\right)^{-1} \quad \text{(A.8)}$$

$$\times\left(\sum_{i=1}^{n}\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(w_i))-\sum_{i=1}^{n-d}\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(z_i)\right.$$

$$\left.+(n-d)\frac{\partial}{\partial\alpha_0}\log(\overline{F}^{\alpha}(s))\right),$$

where, $-A_2/J_{f_{GHCS}}\xrightarrow{P}1$. So, it suffices to show that the numerator is asymptotically $N(0,1)$. Using (A.3) and Slutsky Theorem, we have

$$\frac{\sqrt{n-d}}{\sqrt{n}}\left\{\frac{1}{\sqrt{n-d}}\right.$$

$$\times\left(\sum_{i=1}^{n-d}\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(z_i)-\sum_{i=1}^{n-d}E\left(\frac{\partial}{\partial\alpha_0}\log f^{\alpha}(Z_i)\right)\right) \quad \text{(A.9)}$$

$$\xrightarrow{D}N\left(0,(1-\breve{p})\zeta\right)$$

Now, using Slutsky Theorem, we obtain,

$$\left[\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\partial}{\partial \alpha_0} \log f^{\alpha}(w_i), \quad \frac{1}{\sqrt{n}} \times \sum_{i=1}^{n-d} \left\{ \left(\frac{\partial}{\partial \alpha_0} \log f^{\alpha}(z_i) \right) - E\left(\frac{\partial}{\partial \alpha_0} \log f^{\alpha}(Z_i) \right) \right\} \right],$$

$$\xrightarrow{D} (V, U)$$

where, $V = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{\partial}{\partial \alpha_0} \log f^{\alpha}(w_i) \sim N(0, \wp)$ and $U \sim N(0, (1-\breve{p})\zeta)$ and V and U are independent. Now, using continuous mapping Theorem, (A.8) and (A.9), we conclude that

$$\sum_{i=1}^{n-d} \frac{\partial}{\partial \alpha_0} \log f^{\alpha}(y_i) + (n-d) \frac{\partial}{\partial \alpha_0} \log(\overline{F}(s))$$

$$\xrightarrow{D} N(0, \ \wp + (1-\breve{p})\zeta)$$

and the proof is complete. Thus, we proved that $\sqrt{n}(\hat{\alpha}_n - \alpha_0) = O_p(1)$. So for mis- specified models based on Voung (1989), we can conclude that, $\sqrt{n}(\hat{\alpha}_n - \alpha_*) = O_p(1)$.

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